

# WEIERSTRASS FILTRATION ON TEICHMÜLLER CURVES AND LYAPUNOV EXPONENTS

FEI YU AND KANG ZUO

ABSTRACT. We define Weierstrass filtration for Teichmüller curves. We construct the Harder-Narasimhan filtration of the Hodge bundle of a Teichmüller curve in hyperelliptic loci and low genus non-varying strata, and regain the sum of Lyapunov exponents of Teichmüller curves in these strata.

## CONTENTS

|   |    |
|---|----|
| 1. Introduction   | 1  |
| Acknowledgement   | 3  |
| 2. Background   | 4  |
| 2.1. Moduli spaces  | 4  |
| 2.2. Teichmüller curves   | 4  |
| 2.3. Lyapunov exponents   | 5  |
| 2.4. Vector bundles on curves   | 6  |
| 3. Weierstrass semigroup filtration for one section                         | 6  |
| 3.1. Non-varying Weierstrass semigroups in $\overline{\mathcal{M}}_g(2g-2)$ | 7  |
| 3.2. Weierstrass semigroup filtration in $\overline{\mathcal{M}}_g(2g-2)$   | 8  |
| 3.3. $f_*\mathcal{O}_{aD}(dD)$ and Splitting lemma II                       | 9  |
| 4. Weierstrass filtration for several sections                              | 11 |
| 4.1. Weierstrass filtration   | 11 |
| 4.2. Splitting lemma I and Upper bound lemma                                | 12 |
| 4.3. Application to the sum of Lyapunov exponents                           | 14 |
| 5. Weierstrass exponents  | 15 |
| 5.1. Weierstrass exponents  | 15 |
| 5.2. Hyperelliptic loci   | 17 |
| 5.3. The genus 3 case   | 19 |
| 5.4. The genus 4 case   | 19 |
| 5.5. The genus 5 case   | 20 |
| References  | 22 |

## 1. INTRODUCTION

The computation of Lyapunov exponents is an important subject in the theory of Teichmüller curves. The sum for hyperelliptic connected components is known

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by [9]. In general it is determined by the Siegel-Veech constants which measure the boundary behavior, cf.[10] [6] [15]. In low genus cases [2] [16] [21], the Lyapunov spectrum has been worked out in some partial cases, which is based on a series of work by McMullen [17][18][19]. Some concrete examples like square-tiled surfaces and triangle groups have been computed in [4][8][11].

There are two types of non-varying results of the sum, one for low genus and one for hyperelliptic loci, with two completely different methods of proof [22]. One method uses a translation of the problem into algebraic geometry, in particular slope calculations [6][5], and the other relies on the correspondence to Siegel-Veech constants [9].

In this paper, we will construct Weierstrass filtration of the Hodge bundle based on the dimension of sublinear systems of zeros of holomorphic differentials, define Weierstrass exponents according to Harder-Narasimhan filtration, and compute them in the hyperelliptic loci and low genus non-varying strata, thus give a unified method to compute the sum of Lyapunov exponents.

Let  $g \geq 1$  be an integer, and let  $(m_1, \dots, m_k)$  be a partition of  $2g - 2$ . Denote by  $\Omega\mathcal{M}_g(m_1, \dots, m_k)$  the stratum parameterizing genus  $g$  Riemann surfaces with Abelian differentials that have  $k$  distinct zeros of order  $m_1, \dots, m_k$  respectively.

Let  $C$  be a Teichmüller curve which lies in  $\Omega\mathcal{M}_g(m_1, \dots, m_k)$ . Denote by  $f : S \rightarrow C$  the universal family over a Teichmüller curve with distinct sections  $D_1, \dots, D_k$ . The relative canonical bundle can be computed through the formula (1):

$$\omega_{S/C} \simeq f^* \mathcal{L} \otimes \mathcal{O}(\sum m_i D_i)$$

In hyperelliptic loci and low genus non-varying strata, we will construct the Harder-Narasimhan filtration of  $f_*(\omega_{S/C})$  and show that the factors of the Jordan-Hölder filtration of each semistable graded quotient are line bundles (filtration (10)). Write

$$0 \subset V_1 \subset V_2 \dots \subset V_g = f_*(\omega_{S/C})$$

for the filtration, then  $i$ -th Weierstrass exponents are defined as  $\deg(V_i/V_{i-1})/\deg(\mathcal{L})$  ( $i = 1, \dots, g$ ).

**Theorem 1.1.** *(Theorem 5.5) Let  $C$  be a Teichmüller curve in the hyperelliptic locus of some stratum  $\overline{\Omega\mathcal{M}}_g(m_1, \dots, m_k)$ , and denote by  $(d_1, \dots, d_n)$  the orders of singularities of underlying quadratic differentials. Then the Weierstrass exponents for  $C$  are*

$$1, \{1 - \frac{2k}{d_j + 2}\}_{0 < 2k \leq d_j + 1}$$

This result can be used to regain the sum of Lyapunov exponents in hyperelliptic loci [9]. It was conjectured by Kontsevich and Zorich in [14] (for Teichmüller geodesic flow), and has been shown by M.Bainbridge in the case  $g = 2$  [2]:

$$\sum \lambda_i = \frac{1}{4} \sum_{j \text{ such that } d_j \text{ is odd}} \frac{1}{d_j + 2}.$$

Zorich communicated to D.Chen and M.Möller that, based on a limited number of computer experiments about a decade ago, Kontsevich and Zorich observed that the sum of Lyapunov exponents is non-varying among all the Teichmüller curves in a stratum roughly if the genus plus the number of zeros is less than seven, while the sum varies if this sum is greater than seven. The following two results are entirely

based on the paper [6]. They have shown that the non-varying result of the sum of Lyapunov exponent when  $g \leq 5$  except the strata  $\overline{\Omega\mathcal{M}}_4^{even}(4, 2)$ ,  $\overline{\Omega\mathcal{M}}_4^{odd}(4, 2)$ ,  $\overline{\Omega\mathcal{M}}_5^{odd}(6, 2)$ . The non-varying result is obtained by showing empty intersection of Teichmüller curves with various geometrically defined divisors on moduli spaces of curves. We have made progress in the remaining case with the help of D.Chen.

**Theorem 1.2.** (Theorem 3.5) *A Teichmüller curve in  $\overline{\Omega\mathcal{M}}_g^{hyp}(2g-2)$  has Weierstrass exponents  $\frac{i}{2g-1}$ ,  $i \in G_p = \{1, 3, 5, \dots, 2g-5, 2g-3, 2g-1\}$ .*

*If moreover  $g \leq 5$ , then:*

(2) *A Teichmüller curve in  $\overline{\Omega\mathcal{M}}_g^{odd}(2g-2)$  has Weierstrass exponent  $\frac{i}{2g-1}$ ,  $i \in G_p = \{1, 2, 3, \dots, g-2, g-1, 2g-1\}$ , and  $f_*\omega_{S/C}$  splits into direct sum of line bundles.*

(3) *A Teichmüller curve in  $\overline{\Omega\mathcal{M}}_g^{even}(2g-2)$  has Weierstrass exponent  $\frac{i}{2g-1}$ ,  $i \in G_p = \{1, 2, 3, \dots, g-2, g, 2g-1\}$ .*

**Theorem 1.3.** (Theorem 5.8) *A Teichmüller curve in the strata*

$\overline{\Omega\mathcal{M}}_3(3, 1)$ ,  $\overline{\Omega\mathcal{M}}_3^{odd}(2, 2)$ ,  $\overline{\Omega\mathcal{M}}_3(2, 1, 1)$   
 $\overline{\Omega\mathcal{M}}_4(5, 1)$ ,  $\overline{\Omega\mathcal{M}}_4^{odd}(4, 2)$ ,  $\overline{\Omega\mathcal{M}}_4^{non-hyp}(3, 3)$ ,  $\overline{\Omega\mathcal{M}}_4^{odd}(2, 2, 2)$ ,  $\overline{\Omega\mathcal{M}}_4(3, 2, 1)$   
 $\overline{\Omega\mathcal{M}}_5(5, 3)$ ,  $\overline{\Omega\mathcal{M}}_5^{odd}(6, 2)$

*has explicitly Weierstrass exponents (Table 1, Table 2, Table 3), and  $f_*\omega_{S/C}$  splits into direct sum of line bundles.*

*A Teichmüller curve in the stratum  $\overline{\Omega\mathcal{M}}_4^{even}(4, 2)$  has explicitly Weierstrass exponents.*

A related work about quadratic differentials has been done in [7].

Our basic idea is to construct a filtration of  $f_*\mathcal{O}(\omega_{S/C})$

$$0 \subset \mathcal{L} \subset \dots \subset f_*\mathcal{O}(\omega_{S/C} - \sum d_i D_i) \subset \dots \subset f_*\mathcal{O}(\omega_{S/C})$$

and then compute each graded quotient. But generally, it is difficult to compute the quotient. The quotient is a subbundle of the direct image of a bundle  $\mathcal{O}_{aD_i}(dD_i)$ . We introduce Harder-Narasimhan filtration to study the bundle. The difficulty will disappear if we assume the non-varying of the Weierstrass semigroup of fibers.

The paper is organized as follows. In section 2, we introduce some background material that has appeared in [6]. In section 3, we give a basic example to define Weierstrass semigroup filtration, show that it is the Harder-Narasimhan filtration, and compute the Weierstrass exponent of such Teichmüller curves. In section 4 we define Weierstrass filtration, apply it to compute the sum of Lyapunov exponents. In section 5, we define Weierstrass exponents and compute them in the non-varying sum strata.

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## 2. BACKGROUND

**2.1. Moduli spaces.** Denote by  $\Omega\mathcal{M}_g$  the moduli space of pairs  $(X, \omega)$  where  $X$  is a curve of genus  $g$  and  $\omega$  is a holomorphic one-form on  $X$ . It is fibred over the moduli space  $\mathcal{M}_g$  of curves. Let  $(m_1, \dots, m_k)$  be a partition of  $2g - 2$ , and let  $\Omega\mathcal{M}_g(m_1, \dots, m_k)$  denote the stratum parameterizing one-forms that have  $k$  distinct zeros of order  $m_1, \dots, m_k$  respectively. Denote by  $\Omega\mathcal{M}_g^{hyp}(m_1, \dots, m_k)$  (resp. odd, resp. even) the hyperelliptic (resp. odd theta character, resp. even theta character) connected component. ([15])

Let  $\overline{\mathcal{M}}_g$  denote the Deligne-Mumford compactification of  $\mathcal{M}_g$ . Then  $\Omega\mathcal{M}_g$  extends over  $\overline{\mathcal{M}}_g$ , parameterizing sections of the dualizing sheaf or equivalently stable one-forms. We denote by  $\overline{\Omega\mathcal{M}}_g$  the total space of this extension.

Points in  $\overline{\Omega\mathcal{M}}_g$ , called flat surfaces, are also written as  $(X, \omega)$  with  $\omega$  a stable one-form on  $X$ .

Let  $C$  be a genus  $g$  curve and  $\mathcal{L}$  a line bundle of degree  $d$  on  $C$ . Denote by  $|\mathcal{L}|$  the projective space of one-dimensional subspaces of  $H^0(C, \mathcal{L})$ . For a (projective)  $r$ -dimension linear subspace  $V$  of  $|\mathcal{L}|$ , we call  $(\mathcal{L}, V)$  a linear series of type  $g_d^r$ .

Let  $\underline{\omega} = (\omega_1, \dots, \omega_n)$  be a tuple of integers. The generalized Brill-Noether locus  $BN_{d, \underline{\omega}}^r$  is the locus in  $\overline{\mathcal{M}}_{g, n}$  of pointed curves  $(C, p_1, \dots, p_n)$  with a line bundle  $\mathcal{L}$  of degree  $d$  such that  $\mathcal{L}$  admits a linear system  $g_d^r$  and  $h^0(\mathcal{L}(-\sum w_i p_i)) \geq r$ .

We need the following generalization of Clifford's theorem for stable curves:

**Theorem 2.1.** ([6]) *Let  $C$  be a stable curve and  $D$  an effective divisor with  $\deg(D) \leq 2g - 2$ . Then we have*

$$h^0(\mathcal{O}_C(D)) - 1 \leq \deg(D)/2$$

*if one of the following condition holds:*

- (1)  $C$  is smooth;
- (2)  $C$  has at most two components;
- (3)  $C$  does not have separating nodes and  $\deg(D) \leq 4$ .

**2.2. Teichmüller curves.** A Teichmüller curve  $C$  is an algebraic curve in  $\mathcal{M}_g$  that is totally geodesic with respect to the Teichmüller metric. After suitable base change, we can get a universal family  $f : S \rightarrow C$ , which is a relatively minimal semistable model with disjoint sections  $D_1, \dots, D_k$ ; here  $D_i|_X$  is a zero of  $\omega$  when restrict to each fiber  $X$ . ([6])

Let  $\mathcal{L} \subset f_*\omega_{S/C}$  be the line bundle whose fiber over the point corresponding to  $X$  is  $\mathbb{C}\omega$ , the generating differential of Teichmüller curves; it is also known as the "maximal Higgs" line bundle, in the sense of [23] and [20]. Let  $\Delta \subset \overline{B}$  be the set of points with singular fibers, then the property of being "maximal Higgs" says by definition that  $\mathcal{L} \cong \mathcal{L}^{-1} \otimes \omega_C(\log \Delta)$  and

$$\deg(\mathcal{L}) = (2g(C) - 2 + |\Delta|)/2,$$

together with an identification (relative canonical bundle formula):

$$(1) \quad \omega_{S/C} \simeq f^*\mathcal{L} \otimes \mathcal{O}(\sum m_i D_i)$$

By the adjunction formula we get

$$D_i^2 = -\omega_{S/C} D_i = -m_i D_i^2 - \deg \mathcal{L}$$

and thus

$$(2) \quad D_i^2 = -\frac{1}{m_i + 1} \deg \mathcal{L}$$

The variation of Hodge structures (VHS for short) over a Teichmüller curve decomposes into sub-VHS

$$(3) \quad R^1 f_* \mathbb{C} = (\oplus_{i=1}^r \mathbb{L}_i) \oplus \mathbb{M}$$

Here  $\mathbb{L}_i$  are rank-2 subsystems, maximal Higgs  $\mathbb{L}_i^{1,0} \simeq \mathcal{L}$  for  $i = 1$ , non-unitary but not maximal Higgs for  $i \neq 1$ .[20]

Here we collect some degeneration properties of Teichmüller curves which will be needed in the subsequent sections.

**Theorem 2.2.** [6]

(1) *The section  $\omega$  of the canonical bundle of each smooth fiber over a Teichmüller curve extends to a section  $\omega_\infty$  for each singular fiber  $X_\infty$  over the closure of a Teichmüller curve. The signature of zeros of  $\omega_\infty$  is the same as that of  $\omega$ . Moreover,  $X_\infty$  does not have separating nodes.*

(2) *For Teichmüller curves generated by a flat surface in  $\Omega\mathcal{M}_g(2g-2)$  the degenerating fibers are irreducible.*

(3) *Let  $C$  be a Teichmüller curve generated by an Abelian differential  $(X, \omega)$  in  $\Omega\mathcal{M}_g(\mu)$ . Suppose that an irreducible degenerating fiber  $X_\infty$  over a cusp of  $C$  is hyperelliptic. Then  $X$  is hyperelliptic, hence the whole Teichmüller curve lies in the locus of hyperelliptic flat surfaces.*

(4) *Let  $C$  be a Teichmüller curve generated by a flat surface in  $\Omega\mathcal{M}_5(8)$  even. Then  $C$  does not intersect the Brill-Noether divisor  $BN_3^1$  on  $\mathcal{M}_5$ .*

(5) *Moreover, if  $\mu$  lies in  $\{(4), (3, 1), (6), (5, 1), (3, 3), (3, 2, 1), (8), (5, 3)\}$  and  $(X, \omega)$  is not hyperelliptic, then the non-degenerating fibers of the Teichmüller curve are hyperelliptic.*

**2.3. Lyapunov exponents.** Fix an  $SL_2(\mathbb{R})$ -invariant, ergodic measure  $\mu$  on  $\Omega\mathcal{M}_g$ . Let  $V$  be the restriction of the real Hodge bundle (i.e. the bundle with fibers  $H^1(X, \mathbb{R})$ ) to the support  $M$  of  $\mu$ . Let  $S_t$  be the lift of the geodesic flow to  $V$  via the Gauss-Manin connection. Then Oseledec's multiplicative ergodic theorem guarantees the existence of a filtration

$$0 \subset V_{\lambda_g} \subset \dots \subset V_{\lambda_1} = V$$

by measurable vector subbundles with the property that, for almost all  $m \in M$  and all  $v \in V_m \setminus \{0\}$  one has

$$\|S_t(v)\| = \exp(\lambda_i t + o(t))$$

where  $i$  is the maximal index such that  $v$  is in the fiber of  $V_i$  over  $m$  i.e.  $v \in (V_i)_m$ . The numbers  $\lambda_i$  for  $i = 1, \dots, k \leq \text{rank}(V)$  are called the *Lyapunov exponents* of  $S_t$ . Since  $V$  is symplectic, the spectrum is symmetric in the sense that  $\lambda_{g+k} = -\lambda_{g-k+1}$ . Moreover, from elementary geometric arguments it follows that one always has  $\lambda_1 = 1$ . Thus, the Lyapunov spectrum is defined by the remaining nonnegative Lyapunov exponents

$$\lambda_2 \geq \dots \geq \lambda_g.$$

The bridge between the 'dynamical' definition of Lyapunov exponents and the 'algebraic' method applied in the sequel is given by the following result.

**Theorem 2.3.** ([14][4]) *If the VHS over the Teichmüller curve  $C$  contains a sub-VHS  $\mathbb{W}$  of rank  $2k$ , then the sum of the  $k$  corresponding non-negative Lyapunov*

exponents equals

$$\sum_{i=1}^k \lambda_i^{\mathbb{W}} = \frac{2\deg \mathbb{W}^{(1,0)}}{2g(C) - 2 + |\Delta|}$$

where  $\mathbb{W}^{(1,0)}$  is the  $(1,0)$ -part of the Hodge filtration of the vector bundle associated with  $\mathbb{W}$ . In particular, we have

$$\sum_{i=1}^g \lambda_i = \frac{2\deg f_* \omega_{S/C}}{2g(C) - 2 + |\Delta|}$$

Let  $L(C) = \sum_{i=1}^g \lambda_i$  be the sum of Lyapunov exponents, and put  $k_\mu = \frac{1}{12} \sum_{i=1}^k \frac{m_i(m_i+2)}{m_i+1}$ . Eskin, Kontsevich and Zorich get a formula to compute  $L(C)$  (for the Teichmüller geodesic flow):

**Theorem 2.4.** ([9]) For the VHS over the Teichmüller curve  $C$ , we have

$$L(C) = k_\mu + \frac{\pi^2}{3} c_{\text{area}}(C)$$

where  $c_{\text{area}}(C)$  is the Siegel-Veech constant corresponding to  $C$ .

**2.4. Vector bundles on curves.** The readers are referred to [13] for details about sheaves on algebraic varieties. Let  $C$  be a smooth curve,  $V$  a vector bundle over  $C$  of slope  $\mu(V) := \frac{\deg(V)}{rk(V)}$ . We call  $V$  semistable (resp. stable) if  $\mu(W) \leq \mu(V)$  (resp.  $\mu(W) < \mu(V)$ ) for any subbundle  $W \subset V$ .

A Harder-Narasimhan filtration for  $V$  is an increasing filtration:

$$0 = HN_0(V) \subset HN_1(V) \subset \dots \subset HN_k(V)$$

such that the graded quotients  $gr_i^{HN} = HN_i(V)/HN_{i-1}(V)$  for  $i = 1, \dots, k$  are semistable vector bundles and

$$\mu(gr_1^{HN}) > \mu(gr_2^{HN}) > \dots > \mu(gr_k^{HN})$$

The Harder-Narasimhan filtration is unique.

A Jordan-Hölder filtration for semistable vector bundle  $V$  is a filtration:

$$0 = V_0 \subset V_1 \subset \dots \subset V_k = V$$

such that the graded quotients  $gr_i^V = V_i/V_{i-1}$  are stable of the same slope.

Jordan-Hölder filtration always exist. The graded objects  $gr_i^V = \bigoplus gr_i^V$  does not depend on the choice of the Jordan-Hölder filtration.

### 3. WEIERSTRASS SEMIGROUP FILTRATION FOR ONE SECTION

In this section, we will consider a basic example of Weierstrass semigroup filtration and Weierstrass exponents.

A Teichmüller curve in  $\overline{\Omega\mathcal{M}}_g(2g-2)$  has a relative canonical bundle formula (formula (1)):

$$\omega_{S/C} = f^* \mathcal{L} \otimes \mathcal{O}((2g-2)D).$$

By the projection formula:

$$f_* \omega_{S/C} = \mathcal{L} \otimes f_* \mathcal{O}((2g-2)D)$$

**3.1. Non-varying Weierstrass semigroups in  $\overline{\Omega\mathcal{M}}_g(2g-2)$ .** We will consider the varying of the Weierstrass semigroup of

$$p = D|_F$$

along the fiber  $F$  of universal family on the Teichmüller curve.

**Definition 3.1.** We define Weierstrass semigroup  $H_{p_1, \dots, p_k}$  as follows

$$(4) \quad H_{p_1, \dots, p_k} := \{(n_1, \dots, n_k) | h^0(n_1p_1 + \dots + n_kp_k) = h^0(n_1p_1 + \dots + n_kp_k - p_j) + 1, \forall j\}$$

and we define Weierstrass gap by the formula  $G_{p_1, \dots, p_k} := \mathbb{N}^i - H_{p_1, \dots, p_k}$ .

More information about Weierstrass semigroup of one point  $p$  can be found in [1]. Using the Riemann-Roch theorem, we can compute the cardinality  $G_p$  is equal to  $g$ . In fact:

$$G_p = \{n \in \mathbb{N} : \text{there exists } \omega \in H^0(C, K) \text{ with } \mu_p(\omega) = n - 1\}.$$

This expression is closely related to the filtration we will construct.

We define the weight  $w(H_p)$  of the Weierstrass semigroup  $H_p$  to be:

$$(5) \quad w(H_p) = \sum_{n \in G_p} n - g(g+1)/2$$

which satisfies the inequality

$$(6) \quad w(H_p) \leq g(g-1)/2$$

with equality true if and only if  $2 \in H_p$ .

In fact by [3], generic points in  $\overline{\Omega\mathcal{M}}_g^{odd}(2g-2)$  have Weierstrass gaps  $G_p = \{1, 2, 3, \dots, g-2, g-1, 2g-1\}$ , and generic points in  $\overline{\Omega\mathcal{M}}_g^{even}(2g-2)$  have Weierstrass gaps  $G_p = \{1, 2, 3, \dots, g-2, g, 2g-1\}$ . To compute the Weierstrass exponents and the sum of Lyapunov exponents in both case, we need the nonvarying property of the Weierstrass semigroup in the Teichmüller curve.

**Proposition 3.2.** (1) A Teichmüller curve in  $\overline{\Omega\mathcal{M}}_g^{hyp}(2g-2)$  has Weierstrass gaps  $G_p = \{1, 3, 5, \dots, 2g-5, 2g-3, 2g-1\}$ .

If moreover  $g \leq 5$ , then:

(2) A Teichmüller curve in  $\overline{\Omega\mathcal{M}}_g^{odd}(2g-2)$  has Weierstrass gaps  $G_p = \{1, 2, 3, \dots, g-2, g-1, 2g-1\}$ .

(3) A Teichmüller curve in  $\overline{\Omega\mathcal{M}}_g^{even}(2g-2)$  has Weierstrass gaps  $G_p = \{1, 2, 3, \dots, g-2, g, 2g-1\}$ .

*Proof.* (1) The dimension  $h^0(2p)$  of the fibres of  $f_*\mathcal{O}(2D)$  is an upper-semicontinuous function, and  $h^0(2D|_F) = 2$  at smooth fibres. It is also known that  $2 = \deg(2p) \geq h^0(2p) \geq 2$  for singular fibres and  $H_p$  is a semigroup, so  $\{2, 4, \dots, 2g-4, 2g-2, 2g-1, \dots\} \subset H_p$ . Because  $|G_p| = g$  and we get  $G_p = \mathbb{N} - H_p = \{1, 2, 3, \dots, g-2, g, 2g-1\}$ .

For (2) and (3), by Theorem 2.2, the Teichmüller curve is irreducible and non-hyperelliptic. It does not have separating nodes and  $(g-1) \leq 4$ , so we can use Clifford Theorem 2.1 in each of the two cases:

(2) In  $\overline{\Omega\mathcal{M}}_g^{odd}(2g-2)$  we have  $h^0((g-1)p) \leq 1 + (g-1)/2$ . If  $h^0((g-1)p) = 3$ , the equality implies that it is a hyperelliptic curve, hence a contradiction. So  $h^0((g-1)p) = 1$ .

(3) In  $\overline{\Omega\mathcal{M}}_g^{even}(2g-2)$ , the theta characteristic is even, we have  $h^0((g-1)p) = 2$ . Non-hyperellipticity means that  $h^0((g-2)p) = 1$  for  $g \leq 4$ . It is also true that

$h^0(3p) = 1$  for  $g = 5$  by Theorem 2.2. Using Riemann-Roch theorem we get  $h^0((g+2)p) = h^0((g-2)p) + 1 = 2$ .  $\square$

**3.2. Weierstrass semigroup filtration in  $\overline{\Omega\mathcal{M}}_g(2g-2)$ .** If the Weierstrass semigroup is nonvarying, the following dimensions

$$h(d) := h^0(dD|_F), h^1(dD|_F) = h^0(dD|_F) - d - \chi(\mathcal{O}_F)$$

are constants, so we get vector bundles  $f_*\mathcal{O}(dD), R^1f_*\mathcal{O}(dD)$  by Grauert semicontinuity theorem[12] (in fact,  $f_*\mathcal{O}(dD)$  is always a vector bundle even if Weierstrass semigroup is varying). Define

$$V_{h(d)} := f_*\mathcal{O}(dD) \subset f_*\mathcal{O}((2g-2)D)$$

**Remark 3.3.** *The definition is reasonable: if  $h^0(dD|_F) = h^0((d+1)D|_F)$ , then  $f_*\mathcal{O}(dD) = f_*\mathcal{O}((d+1)D)$  by corollary 4.11.*

Thus we get a filtration of the vector bundle, which we call Weierstrass filtration:

$$0 \subset V_1 \subset V_2 \dots \subset V_g = f_*\mathcal{O}((2g-2)D).$$

Denote by  $d_i$  the  $i$ th element in  $H_p$ .

**Lemma 3.4.** *If the Weierstrass semigroup is nonvarying, then the graded quotient  $V_i/V_{i-1}$  is a line bundle of degree  $\frac{-d_i}{2g-1} \deg \mathcal{L}$ .*

*Proof.* On the surface  $S$  we have the exact sequence:

$$0 \rightarrow \mathcal{O}((d-1)D) \rightarrow \mathcal{O}(dD) \rightarrow \mathcal{O}_D(dD) \rightarrow 0$$

Apply  $f_*$ , and use the fact that  $f$  induces an isomorphism between  $D$  and  $C$  ( $D$  is a section)

$$f_*\mathcal{O}_D(dD) = \mathcal{O}_D(dD)$$

we get

$$\begin{aligned} 0 \rightarrow f_*\mathcal{O}((d-1)D) &\rightarrow f_*\mathcal{O}(dD) \rightarrow \mathcal{O}_D(dD) \xrightarrow{\delta} \\ R^1f_*\mathcal{O}((d-1)D) &\rightarrow R^1f_*\mathcal{O}(dD). \end{aligned}$$

By the nonvarying condition, the two sheaves  $\mathcal{O}_D(dD)$  and  $R^1f_*\mathcal{O}((d-1)D)$  are both locally free. Since subsheaves of a locally free sheaf are locally free, we deduce that  $\text{Ker}(\delta)$  and  $\text{Im}(\delta)$  are both locally free.

For  $d_i \in H_p$  the  $i$ -th element in  $H_p$ ,  $\text{rk } f_*\mathcal{O}((d_i-1)D) = \text{rk } f_*\mathcal{O}(d_iD) - 1$ . Therefore

$$V_i = f_*\mathcal{O}(d_iD), V_{i-1} = f_*\mathcal{O}((d_i-1)D)$$

and  $\text{Ker}(\delta)$  is a line bundle.  $\mathcal{O}_D(d_iD)$  is a line bundle, so  $\text{Im}(\delta) = \mathcal{O}_D(d_iD)/\text{Ker}(\delta)$  is zero. We have:

$$0 \rightarrow f_*\mathcal{O}((d_i-1)D) \rightarrow f_*\mathcal{O}(d_iD) \rightarrow \mathcal{O}_D(d_iD) \rightarrow 0$$

then by the formula (2):

$$\deg(V_i/V_{i-1}) = \deg(f_*\mathcal{O}(d_iD)) - \deg(f_*\mathcal{O}((d_i-1)D)) = d_iD^2 = \frac{-d_i}{2g-1} \deg \mathcal{L}$$

$\square$

We get a filtration of  $f_*\omega_{S/C} = \mathcal{L} \otimes f_*\mathcal{O}((2g-2)D)$ :

$$0 \subset \mathcal{L} \otimes V_1 \subset \mathcal{L} \otimes V_2 \dots \subset \mathcal{L} \otimes V_g = \mathcal{L} \otimes f_*\mathcal{O}((2g-2)D) = f_*\omega_{S/C}$$

We define the  $i$ -th Weierstrass exponent  $w_i$  to be

$$w_i = \deg(\mathcal{L} \otimes V_i / \mathcal{L} \otimes V_{i-1}) / \deg(\mathcal{L}) = 1 - \frac{d_i}{2g-1} = \frac{2g-1-d_i}{2g-1}$$

Then we get

**Theorem 3.5.** *A Teichmüller curve in  $\overline{\Omega\mathcal{M}}_g^{hyp}(2g-2)$  has Weierstrass exponents  $\frac{i}{2g-1}, i \in G_p = \{1, 3, 5, \dots, 2g-5, 2g-3, 2g-1\}$ .*

*If moreover  $g \leq 5$ , then: (2) A Teichmüller curve in  $\overline{\Omega\mathcal{M}}_g^{odd}(2g-2)$  has Weierstrass exponents  $\frac{i}{2g-1}, i \in G_p = \{1, 2, 3, \dots, g-2, g-1, 2g-1\}$ .*

*(3) A Teichmüller curve in  $\overline{\Omega\mathcal{M}}_g^{even}(2g-2)$  has Weierstrass exponents  $\frac{i}{2g-1}, i \in G_p = \{1, 2, 3, \dots, g-2, g, 2g-1\}$ .*

*Proof.* Proposition 3.2 tells us that these Teichmüller curves have non-varying Weierstrass semigroups, and for  $d_i \in H_p$ ,  $2g-1-d_i \in G_p$  we get the result by applying lemma 3.4.  $\square$

Weight formula (5) gives the sum:

$$(7) \quad \sum w_i = \frac{1}{2g-1} \left( \sum_{n \in G_p} n \right) = \frac{1}{2g-1} w(H_p) + \frac{g(g+1)}{2(2g-1)}$$

It has maximal value  $\frac{g^2}{(2g-1)}$  by the inequality (6), where the equality is achieved if and only if  $2 \in H_p$ .

Even if the Weierstrass semigroup is varying, we can also bound  $\deg f_*\omega_{S/C}$  by the sum (7) from the proof of lemma 3.4 since  $\text{im}(\delta)$  is torsion.

**Corollary 3.6.** *If the Weierstrass semigroup is non-varying, then*

$$0 \subset V_1 \subset V_2 \dots \subset V_g = f_*\mathcal{O}((2g-2)D)$$

*is the Harder-Narasimhan filtration.*

*Proof.* Lemma 3.4 tell us that  $\deg(V_i/V_{i-1}) > \deg(V_{i+1}/V_i)$ , we get it by the uniqueness of the Harder-Narasimhan filtration.  $\square$

**3.3.  $f_*\mathcal{O}_{aD}(dD)$  and Splitting lemma II.** We can get more information by analysing the exact sequence

$$(8) \quad 0 \rightarrow f_*\mathcal{O}((d-a)D) \rightarrow f_*\mathcal{O}(dD) \rightarrow f_*\mathcal{O}_{aD}(dD) \xrightarrow{\delta}$$

Here  $\text{Ker}(\delta)$  is controlled by  $f_*\mathcal{O}_{aD}(dD) (= \mathcal{O}_{aD}(dD))$  via the equation (9), which has a good filtration.

**Lemma 3.7.** *The Harder-Narasimhan filtration of  $f_*\mathcal{O}_{aD}(dD)$  is*

$$0 \subset f_*\mathcal{O}_D((d-a+1)D) \dots \subset \dots f_*\mathcal{O}_{(a-1)D}((d-1)D) \subset f_*\mathcal{O}_{aD}(dD)$$

*Proof.* Because  $f$  is an isomorphism between  $D$  and  $C$  ( $D$  is a section),

$$f_*\mathcal{O}_D(jD) = \mathcal{O}_D(jD), R^1 f_*\mathcal{O}_D(jD) = 0$$

From the exact sequence:

$$0 \rightarrow \mathcal{O}_{(i-1)D}((j-1)D) \rightarrow \mathcal{O}_{iD}(jD) \rightarrow \mathcal{O}_D(jD) \rightarrow 0$$

with  $1 \leq i \leq a, d - a + 1 \leq i \leq d$ , we get the long exact sequence:

$$0 \rightarrow f_* \mathcal{O}_{(i-1)D}((j-1)D) \rightarrow f_* \mathcal{O}_{iD}(jD) \rightarrow \mathcal{O}_D(jD) \rightarrow \\ R^1 f_* \mathcal{O}_{(i-1)D}((j-1)D) \rightarrow R^1 f_* \mathcal{O}_{iD}(jD) \rightarrow R^1 f_* \mathcal{O}_D(jD)$$

By induction, we have

$$(9) \quad f_* \mathcal{O}_{iD}(jD) = \mathcal{O}_{iD}(jD), R^1 f_* \mathcal{O}_{iD}(jD) = 0$$

and the exact sequence:

$$0 \rightarrow f_* \mathcal{O}_{(i-1)D}((j-1)D) \rightarrow f_* \mathcal{O}_{iD}(jD) \rightarrow \mathcal{O}_D(jD) \rightarrow 0.$$

Because  $D^2 < 0$ , we get a filtration with strictly decreasing graded quotient line bundles. By the uniqueness of the Harder-Narasimhan filtration, we get the desired result.  $\square$

The filtration can be used to describe the structure of special quotients.

**Lemma 3.8.** (*Splitting lemma II*) *If the dimensions  $h^0(dp) = h^0(d-m)p + m = h^0((d-m-n+1)p) + m$  are non-varying, then  $f_* \mathcal{O}_{aD}$  is split in  $f_* \mathcal{O}_{(a+b)D}$ ,  $a \leq m, b < n$ . In particular, for  $n \geq 2$ ,*

$$f_* \mathcal{O}(dD) / f_* \mathcal{O}((d-m)D) = \bigoplus_{i=0}^{m-1} \mathcal{O}_D((d-i)D).$$

*Proof.* We have the following commutative diagram:

$$\begin{array}{ccccc} f_* \mathcal{O}((d-m)D) & \xhookrightarrow{\quad} & f_* \mathcal{O}((d-m+a)D) & \xrightarrow{\vartheta} & f_* \mathcal{O}_{aD}((d-m+a)D) \\ \theta \parallel \uparrow & & \parallel \uparrow & & \phi \curvearrowleft \uparrow \quad \varphi \uparrow \\ f_* \mathcal{O}((d-m-b)D) & \xhookrightarrow{\quad} & f_* \mathcal{O}((d-m+a)D) & \xrightarrow{\psi} & f_* \mathcal{O}_{(a+b)D}((d-m+a)D) \\ & & & & \uparrow \\ & & & & f_* \mathcal{O}_{bD}((d-m)D) \end{array}$$

Because  $h^0(dp) = h^0(d-m)p + m$ , a similar argument as in Lemma 3.4 implies that  $\vartheta$  is surjective, and  $h^0((d-m)p) = h^0((d-m-n+1)p)$  implies that  $\theta$  is an isomorphism by corollary 4.11.

Thus the image of  $\psi$  is the same as the image of  $\vartheta$ , that is  $f_* \mathcal{O}_{aD}((d-m+a)D)$ . So there is a  $\phi$  with  $\varphi \phi = id$ , hence  $f_* \mathcal{O}_{aD}((d-m+a)D)$  is split in  $f_* \mathcal{O}_{(a+b)D}((d-m+a)D)$ ,  $a \leq m, b < n$ .

If  $n \geq 2$ ,  $f_* \mathcal{O}_{aD}$  is split in  $f_* \mathcal{O}_{(a+1)D}$  for  $a \leq m$ . By induction,  $f_* \mathcal{O}_{mD}$  splits into direct sum of line bundles.

Thus by the uniqueness of Harder-Narasimhan filtration and lemma 3.7, we have

$$f_* \mathcal{O}(dD) / f_* \mathcal{O}((d-m)D) = f_* \mathcal{O}_{mD}(dD) = \bigoplus_{i=0}^{m-1} \mathcal{O}_D((d-i)D).$$

$\square$

**Theorem 3.9.** *For  $g \leq 5$ , the sheaf  $f_*(\omega_{S/C})$  of a Teichmüller curve  $C$  in  $\overline{\Omega\mathcal{M}}_g^{odd}(2g-2)$  splits into direct sum of line bundles.*

*Proof.* Theorem 3.2 gives  $h^0(2(g-1)p) = h^0((g-1)p) + g-1 = h^0((g-2)p) + g-1$ . We have

$$\begin{aligned} f_*\omega_{S/C} &= \mathcal{L} \oplus (\mathcal{L} \otimes (f_*\mathcal{O}((2g-2)D)/f_*\mathcal{O})) \\ &= \mathcal{L} \oplus (\mathcal{L} \otimes (f_*\mathcal{O}((2g-2)D)/f_*\mathcal{O}((g-1)D))) \\ &= \mathcal{L} \oplus (\bigoplus_{i=0}^{g-1} \mathcal{O}_D((2g-2-i)D) \otimes \mathcal{L}) \end{aligned}$$

here the first equality is by Equation (3) and the last equality follows from Lemma 3.8.  $\square$

#### 4. WEIERSTRASS FILTRATION FOR SEVERAL SECTIONS

In this section we will define three kinds of filtration: Weierstrass filtration, Weierstrass semigroup filtration and Weierstrass pair filtration. The first one together with the upper bound lemma 4.10 is used to get coarse information about the upper bound of the sum. The second and the third one can be used to get more precise information about each quotient.

**4.1. Weierstrass filtration.** From the exact sequence

$$0 \rightarrow f_*\mathcal{O}(d_1D_1 + \dots + d_kD_k) \rightarrow f_*\mathcal{O}(m_1D_1 + \dots + m_kD_k) = f_*(\omega_{S/C}) \otimes \mathcal{L}^{-1}$$

and the fact that all subsheaves of a locally free sheaf on a curve are locally free, we deduce that  $f_*\mathcal{O}(d_1D_1 + \dots + d_kD_k)$  is a vector bundle of rank  $h^0(d_1p_1 + \dots + d_kp_k)$ , here  $p_i = D_i|_F$ ,  $F$  is a generic fiber.

For  $1 \leq i \leq g$ , we define

$$\begin{aligned} W_i &= \{(d_1, \dots, d_k) | h^0(d_1p_1 + \dots + d_kp_k) = i \text{ for general fiber}\} \\ WS_i &= \{(d_1, \dots, d_k) | h^0(d_1p_1 + \dots + d_kp_k) = i \text{ non varying}\} \end{aligned}$$

For each element in  $(d_1, \dots, d_k) \in W_i$ ,  $f_*\mathcal{O}(d_1D_i + \dots + d_kD_k)$  is a rank  $i$  vector bundle.

For each element in  $(d_1, \dots, d_k) \in WS_i$ ,  $f_*\mathcal{O}(d_1D_i + \dots + d_kD_k)$  is a rank  $i$  vector bundle. By non varying condition,  $R^1f_*\mathcal{O}(d_1D_i + \dots + d_kD_k)$  is also a vector bundle.

We also define  $(d_1, \dots, d_k) \leq (d'_1, \dots, d'_k)$  if  $d_i \leq d'_i$  for all  $i$ .  $(d_1, \dots, d_k) < (d'_1, \dots, d'_k)$  if  $d_i \leq d'_i$  but not all  $d_i = d'_i$ .

**Definition 4.1.** We define the set of Weierstrass filtration as follows:

$$WF = \{\{(d_1, \dots, d_k)_i\} | \text{at most one } (d_1, \dots, d_k)_i \in W_i, (d_1, \dots, d_k)_i < (d_1, \dots, d_k)_j, \text{if } i < j\}$$

An element  $\{(d_1, \dots, d_k)_i\} \in WF$  is a filtration of vector bundles of  $f_*\mathcal{O}(m_1D_1 + \dots + m_kD_k)$ .

$$0 \subset \dots \subset f_*\mathcal{O}((d_1, \dots, d_k)_i) \subset \dots \subset f_*\mathcal{O}(m_1D_1 + \dots + m_kD_k)$$

Because in general  $R^1f_*\mathcal{O}(d_1D_i + \dots + d_kD_k)$  is not locally free, the filtration won't give us the desired properties to compute its degree. So in many cases, we need the non-varying condition to get more information.

Denote by  $\underline{d}$  the tuple  $(d_1, \dots, d_k)$

**Definition 4.2.** We define the set of Weierstrass semigroup filtration as follows:

$$WSF = \{\{\underline{d}_i\} | \text{there is only one } \underline{d}_i \text{ from } WS_i \cap H_{p_1, \dots, p_k}, \text{ and } \underline{d}_i < \underline{d}_{i+1}\}$$

Assuming the Weierstrass semigroup to be non-varying, we will use the Weierstrass semigroup filtration to define Weierstrass exponents in the next section.

**Example 4.3.** *There are many Weierstrass semigroup filtration for a Teichmüller curve in the hyperelliptic locus by proposition 5.4.*

Under some weak assumptions, the following filtration is also useful for computational and theoretical reasons.

**Definition 4.4.** *We define the set of Weierstrass pair filtration as follows:*

$$WPF = \{\{\underline{d}_i, \underline{d}'_{i+1}\} \mid \underline{d}_i \in WS_i, \underline{d}'_i \in W_i, \underline{d}'_i < \underline{d}_i, \underline{d}'_{i+1} = \underline{d}_i + (0, \dots, 1, \dots, 0)\}$$

To define these we need to verify that the exact sequence

$$\begin{aligned} 0 \rightarrow f_* \mathcal{O}(\underline{d}_i) &\rightarrow f_* \mathcal{O}(\underline{d}'_{i+1}) \rightarrow \mathcal{O}_{\underline{d}'_{i+1} - \underline{d}_i}(\underline{d}'_{i+1}) \xrightarrow{\delta} \\ R^1 f_* \mathcal{O}(\underline{d}_i) &\rightarrow R^1 f_* \mathcal{O}(\underline{d}'_{i+1}) \end{aligned}$$

are such that  $rk f_* \mathcal{O}(\underline{d}_i) + 1 = rk f_* \mathcal{O}(\underline{d}'_{i+1})$ ,  $\mathcal{O}_{\underline{d}'_{i+1} - \underline{d}_i}(\underline{d}'_{i+1})$  is a line bundle and that  $R^1 f_* \mathcal{O}(\underline{d}_i)$  is locally free. Note also that  $\underline{d}'_i < \underline{d}_i$  implies  $f_* \mathcal{O}(\underline{d}'_i) = f_* \mathcal{O}(\underline{d}_i)$ . Thus an element  $\{\underline{d}_i, \underline{d}'_{i+1}\} \in WPF$  is a filtration of vector bundles

$$0 \subset \dots \subset f_* \mathcal{O}(\underline{d}_i) \subset f_* \mathcal{O}(\underline{d}'_{i+1}) = f_* \mathcal{O}(\underline{d}_{i+1}) \subset \dots \subset f_* \mathcal{O}(m_1 D_1 + \dots + m_k D_k)$$

**Example 4.5.** *There is a Weierstrass pair filtration (11) in the proof of the stratum  $\overline{\Omega\mathcal{M}}_4^{even}(4, 2)$ .*

**4.2. Splitting lemma I and Upper bound lemma.** The next lemma describes a splitting structure of the quotient:

**Lemma 4.6** (Splitting lemma I). *If  $h^0(\sum d_i p_i) = h^0(\sum (d_i - a_i)p_i) + \sum a_i$  is non-varying, then*

$$f_* \mathcal{O}(\sum d_i D_i) / f_* \mathcal{O}(\sum (d_i - a_i) D_i) = \bigoplus f_* \mathcal{O}_{a_i D_i}(d_i D_i)$$

*Proof.* From the exact sequence

$$0 \rightarrow \mathcal{O}(\sum (d_i - a_i) D_i) \rightarrow \mathcal{O}(\sum d_i D_i) \rightarrow \mathcal{O}_{\sum a_i D_i}(\sum d_i D_i) \rightarrow 0$$

we get the long exact sequence

$$\begin{aligned} 0 \rightarrow f_* \mathcal{O}(\sum (d_i - a_i) D_i) &\rightarrow f_* \mathcal{O}(\sum d_i D_i) \rightarrow f_* \mathcal{O}_{\sum a_i D_i}(\sum d_i D_i) \xrightarrow{\delta} \\ R^1 f_* \mathcal{O}(\sum (d_i - a_i) D_i) &\rightarrow R^1 f_* \mathcal{O}(\sum d_i D_i) \rightarrow 0 \end{aligned}$$

Because  $Ker(\delta)$  and  $Im(\delta)$  are both locally free and because

$$rk(R^1 f_* \mathcal{O}(\sum (d_i - a_i) D_i)) = rk(R^1 f_* \mathcal{O}(\sum d_i D_i))$$

we get

$$0 \rightarrow f_* \mathcal{O}(\sum (d_i - a_i) D_i) \rightarrow f_* \mathcal{O}(\sum d_i D_i) \rightarrow f_* \mathcal{O}_{\sum a_i D_i}(\sum d_i D_i) \rightarrow 0$$

Since  $D_i D_j = 0$  for  $i \neq j$ , we have

$$f_* \mathcal{O}_{\sum a_i D_i}(\sum d_i D_i) = f_*(\bigoplus \mathcal{O}_{a_i D_i}(\sum d_i D_i)) = f_*(\bigoplus \mathcal{O}_{a_i D_i}(d_i D_i)) = \bigoplus f_* \mathcal{O}_{a_i D_i}(d_i D_i)$$

□

We often uses it with lemma 3.8.

**Corollary 4.7.** *For a Teichmüller curve in  $\overline{\Omega\mathcal{M}}_g(m_1, \dots, m_k)$  with theta characteristic  $h^0((m_1/2, \dots, m_k/2)) = 1$ , the sheaf  $f_* \omega_{S/C}$  splits into direct sum of line bundles.*

*Proof.* Because

$$h^0\left(\sum m_i p_i\right) = h^0\left(\sum \frac{m_i}{2} p_i\right) + g - 1 = h^0\left(\sum \frac{m_i}{2} p_i\right) + \sum \frac{m_i}{2}$$

we can apply lemma 4.6. Because

$$h^0\left(\left(\sum \frac{m_i}{2} p_i\right) + p_j\right) - 1 = h^0\left(\sum \frac{m_i}{2} p_i\right) = 1 = h^0\left(\left(\sum \frac{m_i}{2} p_i\right) - p_j\right)$$

it suffices to apply lemma 3.8 to each  $D_j$ .  $\square$

**Corollary 4.8.** *Each graded quotient of a filtration in  $WSF$  resp. in  $WPF$  is a line bundle with computable degree. So the sum can be computed. All filtration in  $WSF$  have the same sum  $\deg f_* \omega_{S/C} - \text{gdeg}(\mathcal{L})$ .*

*Proof.* For any filtration in  $WSF$ , we have the exact sequence

$$WSF : 0 \rightarrow f_* \mathcal{O}(\underline{d}_i - D_j) \rightarrow f_* \mathcal{O}(\underline{d}_i) \rightarrow \mathcal{O}_{D_j}(\underline{d}_i) \rightarrow 0$$

where  $\underline{d}_{i-1} \leq \underline{d}_i - D_j$ .

Similarly for  $WPF$  we have the exact sequence

$$WPF : 0 \rightarrow f_* \mathcal{O}(\underline{d}_i) \rightarrow f_* \mathcal{O}(\underline{d}'_{i+1}) \rightarrow \mathcal{O}_{\underline{d}'_{i+1} - \underline{d}_i}(\underline{d}'_{i+1}) \rightarrow 0$$

Therefore each graded quotient of the filtration is a line bundle  $\mathcal{O}_{D_j}(\underline{d}_i)$  (resp.  $\mathcal{O}_{\underline{d}'_{i+1} - \underline{d}_i}(\underline{d}'_{i+1})$ ), whose degree is computable.

For the second part of the claim, the sum is

$$\deg f_* \mathcal{O}(m_1 D_1 + \dots + m_k D_k) = \deg f_* \omega_{S/C} - \text{gdeg}(\mathcal{L})$$

$\square$

**Example 4.9.** *An explicit formula about the sum has been got in corollary 5.6 for a Teichmüller curve in the hyperelliptic locus.*

When  $h^0(\sum d_i p_i) \leq h^0(\sum (d_i - a_i) p_i) + \sum a_i$ , we can get an upper bound for the quotient by the properties of the Harder-Narasimhan filtration, even if  $h^0(\sum d_i p_i), h^0(\sum (d_i - a_i) p_i)$  is varying.

**Lemma 4.10** (Upper bound lemma). *The degree  $\deg(f_* \mathcal{O}(\sum d_i D_i) / f_* \mathcal{O}(\sum (d_i - a_i) D_i))$  is smaller than the maximal sums of  $h^0(\sum d_i p_i) - h^0(\sum (d_i - a_i) p_i)$  line bundles in*

$$\bigcup_i \bigcup_{j=0}^{a_i-1} \mathcal{O}_{D_i}((d_i - j) D_i)$$

(here  $p_i = D_i|_F$ ,  $F$  being a general fiber).

*Proof.* By lemma 3.7, the graded sum of the Harder-Narasimhan filtration of  $\bigoplus_i f_* \mathcal{O}_{a_i D_i}(d_i D_i)$  is

$$\text{grad}(HN(\bigoplus_i f_* \mathcal{O}_{a_i D_i}(d_i D_i))) = \bigoplus_i \bigoplus_{j=0}^{a_i-1} \mathcal{O}_{D_i}((d_i - j) D_i)$$

In the proof of Splitting lemma 4.6 the kernel  $\ker(\delta)$  is a subbundle of rank equal to  $h^0(\sum d_i p_i) - h^0(\sum (d_i - a_i) p_i)$ :

$$f_* \mathcal{O}(\sum d_i D_i) / f_* \mathcal{O}(\sum (d_i - a_i) D_i) \subset \bigoplus_i f_* \mathcal{O}_{a_i D_i}(d_i D_i)$$

We get the result by using the maximality of Harder-Narasimhan polygon: each rank  $r$  subbundle has degree smaller than the maximal sum of  $r$  line bundles in

$$\bigcup_i \bigcup_{j=0}^{a_i-1} \mathcal{O}_{D_i}((d_i - j)D_i)$$

□

**Corollary 4.11.** *If  $h^0(\sum d_i p_i) = h^0(\sum (d_i - a_i) p_i)$  holds in a general fiber, then we have the equality  $f_* \mathcal{O}(\sum d_i D_i) = f_* \mathcal{O}(\sum (d_i - a_i) D_i)$ .*

**4.3. Application to the sum of Lyapunov exponents.** The existence of Weierstrass semigroup (pair) filtration is convenient for computation.

**Corollary 4.12.** *If there exists a Weierstrass semigroup (pair) filtration for a Teichmüller curve, then we can compute the sum of Lyapunov exponents. The denominator of the sum of the Lyapunov exponents divides  $(m_1 + 1) \dots (m_k + 1)$*

*Proof.* The sum of Lyapunov exponents is  $L(C) = \deg f_* \omega_{S/C} / \deg(\mathcal{L})$  by Theorem 2.3. The sum of Weierstrass semigroup (pair) filtration is  $\deg f_* \omega_{S/C} - g\deg(\mathcal{L})$  by corollary 4.8.

Because  $D_i^2 / \deg \mathcal{L} = -\frac{1}{m_i+1}$ , each quotient has denominator  $(m_i + 1)$ , we get the second claim. □

In many cases, because of the absence of Weierstrass semigroup (pair) filtration, we can not get more precise information about  $f_* \omega_{S/C}$ . But the partial filtration is enough to give some upper bound of it. Use this coarse filtration, we have:

**Corollary 4.13.** *The sum of Lyapunov exponents satisfies the inequality*

$$L(C) \leq \frac{3g}{4} - \frac{1}{8}(-2 + \sum_{m_i \text{ even}} \frac{m_i}{m_i+1} + \sum_{m_i \text{ odd}} 1)$$

*Proof.* There is a rank  $g - 1$  subbundle

$$f_* \mathcal{O}(m_1 D_1 + \dots + m_k D_k) / \mathcal{O}_C \subset \bigoplus_i f_* \mathcal{O}_{m_i D_i}(m_i D_i)$$

It is obvious by lemma 3.7 that  $\text{grad}(HN(\bigoplus_i f_* \mathcal{O}_{m_i D_i}(m_i D_i))) = \bigoplus_i \bigoplus_{j=1}^{m_i} \mathcal{O}_{D_i}(j D_i)$ . By lemma 4.10, we want to bound the maximum of sums of  $g - 1$  line bundles in  $\bigcup_i \bigcup_{j=1}^{m_i} \mathcal{O}_{D_i}(j D_i)$ . The maximum of sums of  $n_i$  line bundles in  $\bigcup_{j=1}^{m_i} \mathcal{O}_{D_i}(j D_i)$  is reached by  $\bigcup_{j=1}^{n_i} \mathcal{O}_{D_i}(j D_i)$ , and the sum is  $(-\frac{n_i(n_i+1)}{2(m_i+1)}) \deg \mathcal{L}$ . We get

$$\deg(f_* \mathcal{O}(m_1 D_1 + \dots + m_k D_k) / \mathcal{O}_C) \leq \max_{\sum n_i = g-1} \left\{ -\sum \frac{n_i(n_i+1)}{2(m_i+1)} \right\} \deg \mathcal{L}$$

Because  $\sum n_i = g - 1 = (\sum m_i)/2$ , there must be some  $n_j$  such that  $n_j \leq (m_j - 1)/2$  if  $n_i > (m_i + 1)/2$ . From the inequality

$$\deg(\mathcal{O}_{D_i}(n_i D_i)) < -\frac{1}{2} \deg \mathcal{L} \leq \deg(\mathcal{O}_{D_j}((n_j + 1) D_j))$$

there must be some  $n_i$  such that  $n_i \geq (m_i + 1)/2$  if  $n_j < (m_j - 1)/2$ , and

$$\deg(\mathcal{O}_{D_i}(n_i D_i)) \leq -\frac{1}{2} \deg \mathcal{L} < \deg(\mathcal{O}_{D_j}((n_j + 1) D_j))$$

We can sum up the value  $-\sum \frac{n_i(n_i+1)}{2(m_i+1)}$  by changing  $n_i$  to  $n_i - 1$  and  $n_j$  to  $n_j + 1$  in both case.

So we assume that  $n_i$  be equal to  $m_i/2$  when  $m_i$  is even, and  $n_i = (m_i - 1)/2$  or  $(m_i + 1)/2$  when  $m_i$  is odd, with the property  $\#\{n_i = (m_i - 1)/2\} = k = \#\{n_i = (m_i + 1)/2\}$ .

$$\begin{aligned} & \sum_{m_i \text{ even}} \frac{m_i(m_i+2)}{8(m_i+1)} + \sum_{n_i=(m_i-1)/2} \frac{(m_i-1)(m_i+1)}{8(m_i+1)} + \sum_{n_i=(m_i+1)/2} \frac{(m_i+1)(m_i+3)}{8(m_i+1)} \\ &= \sum_{m_i \text{ even}} \left( \frac{m_i}{8} + \frac{m_i}{8(m_i+1)} \right) + \sum_{n_i=(m_i-1)/2} \frac{m_i-1}{8} + \sum_{n_i=(m_i+1)/2} \frac{m_i+3}{8} \\ &= \sum_{m_i \text{ even}} \frac{m_i}{8} + \sum_{m_i \text{ even}} \frac{m_i}{8(m_i+1)} + \frac{(-k)}{8} + \frac{3k}{8} \\ &= \frac{2g}{8} + \frac{1}{8} \left( -2 + \sum_{m_i \text{ even}} \frac{m_i}{m_i+1} + \sum_{m_i \text{ odd}} 1 \right) \end{aligned}$$

so we get

$$L(C) \leq \frac{3g}{4} - \frac{1}{8} \left( -2 + \sum_{m_i \text{ even}} \frac{m_i}{m_i+1} + \sum_{m_i \text{ odd}} 1 \right)$$

□

**Remark 4.14.** *D.W. Chen and M.Möller [22] have got a bound by using Cornalba-Harris-Xiao's slope inequality [24]:*

$$L(C) \leq \frac{3g}{(g-1)} \kappa_\mu = \frac{g}{4(g-1)} \sum_{i=1}^k \frac{m_i(m_i+2)}{m_i+1} \leq \frac{3g}{4}$$

## 5. WEIERSTRASS EXPONENTS

This section is devoted to the construction of the Harder-Narasimhan filtration of  $f_*\mathcal{O}(m_1D_1 + \dots + m_kD_k)$  and the definition of Weierstrass exponents under some additional assumptions.

**5.1. Weierstrass exponents.** If there is a filtration

$$(10) \quad 0 \subset V_1 \subset V_2 \dots \subset V_g = f_*\mathcal{O}(m_1D_1 + \dots + m_kD_k)$$

satisfying: (1)  $V_i/V_{i-1}$  is a line bundle, (2)  $\deg(V_i/V_{i-1})$  is decreasing in  $i$ ,

then it is the Harder-Narasimhan filtration of  $f_*\mathcal{O}(m_1D_1 + \dots + m_kD_k)$ , because each graded quotient  $V_i/V_{i-1}$  is already stable as it is a line bundle. The factors  $gr_j^{gr_i^{HN}}$  of the Jordan-Hölder filtration of each semistable graded quotient  $gr_i^{HN}$  are line bundles.

We also get a filtration for  $f_*\omega_{S/C}$

$$0 \subset \mathcal{L} \otimes V_1 \subset \mathcal{L} \otimes V_2 \dots \subset \mathcal{L} \otimes V_g = \mathcal{L} \otimes f_*\mathcal{O}(\sum m_i D_i) = f_*\omega_{S/C}$$

**Definition 5.1.** *If there exists a filtration as in (10), we define  $i$ -th Weierstrass exponent  $w_i$  as follows:*

$$w_i = \deg(\mathcal{L} \otimes V_i / \mathcal{L} \otimes V_{i-1}) / \deg(\mathcal{L}).$$

When  $H_{p_1, \dots, p_k} \in \cup_i WS_i$  is non-varying, there are many Weierstrass semigroup filtration, and we can construct the Harder-Narasimhan filtration recursively.

**Theorem 5.2.** *Assume that the Weierstrass semigroup is non-varying, then we can construct a filtration*

$$0 \subset V_1 \subset V_2 \dots \subset V_g = f_* \mathcal{O}(m_1 D_1 + \dots + m_k D_k)$$

satisfying:

- (1)  $V_i/V_{i-1}$  is a line bundle,
- (2)  $\deg(V_i/V_{i-1})$  is decreasing in  $i$ .

*Proof.* For every Weierstrass semigroup element  $(d_1, \dots, d_k)$  of a general fiber, we define the length of the set to be

$$l(d_1, \dots, d_k) = \min\{-d_1/(m_1 + 1), \dots, -d_k/(m_k + 1)\},$$

where the fraction  $-d_j/(m_j + 1)$  is equal to  $\deg \mathcal{O}_{D_j}(d_j D_j)/\deg(\mathcal{L})$ .

For any vector bundle of the form  $f_* \mathcal{O}(a_1 D_1 + \dots + a_k D_k)$ , we define the following set

$$L(a_1, \dots, a_k) := \{\underline{d} \in H_{p_1, \dots, p_k} \mid f_* \mathcal{O}(d_1 D_1 + \dots + d_k D_k) = f_* \mathcal{O}(a_1 D_1 + \dots + a_k D_k), \underline{d} \leq \underline{a}\}$$

It is not empty because it contains the element  $(d_1, \dots, d_k) \in H_{p_1, \dots, p_k}$ , for which the sum  $\sum_{i=1}^k d_i$  reaches the minimum when  $(d_1, \dots, d_k)$  varies in

$$\{\underline{d} \mid f_* \mathcal{O}(d_1 D_1 + \dots + d_k D_k) = f_* \mathcal{O}(a_1 D_1 + \dots + a_k D_k), \underline{d} \leq \underline{a}\}.$$

We then construct the set  $L_i$  and define the number  $l_i$  recursively:

$$L_g = \{(m_1, \dots, m_k)\}, l_g = l(m_1, \dots, m_k)$$

.....

$$L_i = \{L(d_1, \dots, d_j - 1, \dots, d_k) \mid (d_1, \dots, d_k) \in L_{i+1}, -d_j/(m_j + 1) = l_{i+1}\}$$

$$l_i = \min\{l(d_1, \dots, d_k) \mid (d_1, \dots, d_k) \in L_i\}$$

.....

If  $L_i$  is non empty, then  $l_i$  is defined. If  $l_i$  is defined, then  $L_{i-1}$  is non empty because  $L(d_1, \dots, d_j - 1, \dots, d_k)$  is non empty ( $i \geq 2$ ). So the definition makes sense.

It is obvious that  $\text{rk}(f_* \mathcal{O}(d_1 D_1 + \dots + d_k D_k)) = i$  for any  $(d_1, \dots, d_k) \in L_i$ .

For any  $(e_1, \dots, e_k) \in L_{i-1}$ , by our construction, there is a  $(d_1, \dots, d_k) \in L_i$ ,  $-d_j/(m_j + 1) = l_i$  such that  $(e_1, \dots, e_k)$  lies in  $L(d_1, \dots, d_j - 1, \dots, d_k)$ . We repeat the process from  $L_1$ , then inductively we obtain a filtration

$$0 \subset V_1 \subset V_2 \dots \subset V_g = f_* \mathcal{O}(m_1 D_1 + \dots + m_k D_k)$$

with  $V_i/V_{i-1}$  being a line bundle. The filtration is not unique because there maybe many chooses in each step.

From the equalities  $V_i = f_* \mathcal{O}(d_1 D_1 + \dots + d_k D_k)$

$$V_{i-1} = f_* \mathcal{O}(\sum e_i D_i) = f_* \mathcal{O}(d_1 D_1 + \dots + (d_j - 1) D_j + \dots + d_k D_k)$$

and the exact sequence

$$0 \rightarrow V_{i-1} \rightarrow V_i \rightarrow \mathcal{O}_{D_j}(d_j D_j)$$

we get

$$\deg(V_i/V_{i-1})/\deg(\mathcal{L}) \leq \deg \mathcal{O}_{D_j}(d_j D_j)/\deg(\mathcal{L}) = -d_j/(m_j + 1) = l_i$$

and

$$\begin{aligned}
l_{i-1} &\geq \min\{-e_1/(m_1+1), \dots, -e_k/(m_k+1)\} \\
&\geq \min\{-d_1/(m_1+1), \dots, -(d_j-1)/(m_j+1), \dots, -d_k/(m_k+1)\} \\
&\geq \min\{-d_1/(m_1+1), \dots, -d_j/(m_j+1), \dots, -d_k/(m_k+1)\} \\
&= -d_j/(m_j+1) = l_i
\end{aligned}$$

When we assume the Weierstrass semigroup to be non-varying, we get  $\deg(V_i/V_{i-1})/\deg(\mathcal{L}) = l_i$  by lemma 4.6. Therefore  $\deg(V_i/V_{i-1})$  is decreasing in  $i$ .  $\square$

**Remark 5.3.** *From the proof, we can see that the Harder-Narasimhan filtration is constructed under the weak assumption that a subset of the Weierstrass semigroup is non-varying. This fact will be verified in hyperelliptic loci and non-varying strata of genus less than or equal five.*

**5.2. Hyperelliptic loci.** The square of any holomorphic 1-form  $\omega$  on a hyperelliptic curve fiber  $F$  is a pullback  $(\omega)^2 = p^*q$  of some meromorphic quadratic differential  $q$  with simple poles on  $\mathbb{P}^1$  where the projection  $p : F \rightarrow \mathbb{P}^1$  is the quotient over the hyperelliptic involution. Denote by  $\mathcal{Q}(d_1, \dots, d_n)$  the stratification by orders of zeros of the corresponding quadratic differentials (see [9] for more details).

**Proposition 5.4.** *Let  $C$  be a Teichmüller curve in the hyperelliptic locus of some stratum  $\overline{\Omega\mathcal{M}}_g(m_1, \dots, m_k)$ , and denote by  $(d_1, \dots, d_n)$  the orders of singularities of the corresponding quadratic differentials. Then there exists a filtration*

$$0 \subset V_1 \subset V_2 \dots \subset V_g = f_*\mathcal{O}(m_1D_1 + \dots + m_kD_k)$$

satisfying: (1)  $V_i/V_{i-1}$  is a line bundle for each  $i$ , (2)  $\deg(V_i/V_{i-1})$  is decreasing in  $i$ .

*Proof.* Let  $F$  be the covering flat surface belonging to the stratum  $\overline{\Omega\mathcal{M}}_g(m_1, \dots, m_k)$ , then the resulting holomorphic form  $\omega$  on  $F$  has zeros of the following degrees:

A zeros  $p_j$ , which is an order  $d$ , of meromorphic quadratic differentials  $q$  on  $\mathbb{P}^1$  gives rise to zeros on  $F([9])$ :

- (1) Two zeros  $p_{1j}, p_{2j}$  of degree  $m = d/2$ , when  $d$  is even. In this case,  $p_{1j}, p_{2j}$  is a  $g_2^1$ , i.e.  $h^0(p_{1j} + p_{2j}) = 2$ .
- (2) One zero  $q_j$  of degree  $m = d + 1$ , when  $d > 0$  is odd. In this case,  $q_j$  is a Weierstrass point, i.e.  $h^0(2q_j) = 2$ .

Denote by  $Q_j$  resp.  $P_{1j}$  resp.  $P_{2j}$  the section containing  $q_j$  resp.  $p_{1j}$  resp.  $p_{2j}$

For each fiber  $F$ , the Weierstrass semigroup has at least a non-varying subgroup generated by the elements  $\{\{2q_j\}_{d_j \text{ odd}}, \{p_{1j} + p_{2j}\}_{d_j \text{ even}}\}$ , which is equal to:

$$\left\{ \sum_{d_j \text{ odd}} 2k_j q_j + \sum_{d_j \text{ even}} n_j (p_{1j} + p_{2j}) \mid 2k_j \leq d_j + 1, 2n_j \leq d_j + 1 \right\}$$

$(n_j \leq d_j/2 \Leftrightarrow 2n_j \leq d_j + 1 \text{ for } d_j \text{ even})$

We order the following  $g-1$  numbers  $\{\{-\frac{2k}{d_j+2}\}_{2k \leq d_j+1}\}$  to  $\{N_1, \dots, N_{g-1}\}$  in decreasing order.

By the rule,  $-\frac{2k}{d_j+2} \rightarrow 2Q_j$ , when  $d_j$  is odd; and  $-\frac{2k}{d_j+2} \rightarrow (P_{1j} + P_{2j})$ , when  $d_j$  is even. We transform  $\{N_1, \dots, N_{g-1}\}$  to a new symbol set with  $g-1$  element  $\{T_1, \dots, T_{g-1}\}$ .

Let  $V_i = f_*\mathcal{O}(T_1 + \dots + T_{i-1})$ , then we get a filtration

$$0 \subset V_1 \subset V_2 \dots \subset V_g = f_*\mathcal{O}(m_1D_1 + \dots + m_kD_k).$$

If  $N_i = -\frac{2k}{d_j+2}$  and  $d_j$  is odd, then  $\{N_1, \dots, N_i\}$  contains  $-\frac{2l}{d_j+2}$ ,  $l \leq k$ , hence  $T_1 + \dots + T_{i-1}$  contain  $2kQ_j$ .

Since  $V_{i-1} = f_*\mathcal{O}(T_1 + \dots + T_{i-1}) = f_*\mathcal{O}(T_1 + \dots + T_{i-1} + Q_j)$ , by the non-varying property and Lemma 4.6, we get  $V_i/V_{i-1} = \mathcal{O}_{Q_j}(2kQ_j)$ ,

$$\deg(V_i/V_{i-1}) = \deg(\mathcal{O}_{Q_j}(2kQ_j)) = -\frac{2k}{(d_j+1)+1} = N_i.$$

If  $N_i = -\frac{2k}{d_j+2}$  and  $d_j$  is even, then  $\{N_1, \dots, N_i\}$  just contains  $-\frac{2l}{d_j+2}$ ,  $l \leq k$ .  $T_1 + \dots + T_{i-1}$  just contains  $k(P_{1j} + P_{2j})$ .

Similarly, we have  $V_{i-1} = f_*\mathcal{O}(T_1 + \dots + T_{i-1}) = f_*\mathcal{O}(T_1 + \dots + T_{i-1} + P_{1j})$ , and by the non-varying property and Lemma 4.6, we get  $V_i/V_{i-1} = \mathcal{O}_{P_{2j}}(k(P_{1j} + P_{2j}))$ , and that

$$\deg(V_i/V_{i-1}) = \deg(\mathcal{O}_{P_{2j}}(k(P_{1j} + P_{2j}))) = -\frac{k}{d_j/2+1} = N_i.$$

So  $\deg(V_i/V_{i-1})$  is decreasing in  $i$ . □

**Theorem 5.5.** *Let  $C$  be a Teichmüller curve in the hyperelliptic locus of some stratum  $\overline{\mathcal{M}}_g(m_1, \dots, m_k)$ , and denote by  $(d_1, \dots, d_n)$  the orders of singularities of underlying quadratic differentials. Then  $C$  has Weierstrass exponents*

$$1, \{1 - \frac{2k}{d_j+2}\}_{0 < 2k \leq d_j+1}$$

*Proof.* The  $i$ -th Weierstrass exponents as follows:

$$w_1 = 1, w_i = \deg(\mathcal{L} \otimes V_i / \mathcal{L} \otimes V_{i-1}) / \deg(\mathcal{L}) = 1 + N_{i-1}.$$

□

**Corollary 5.6** (cf.[9]). *Let  $C$  be a Teichmüller curve in the hyperelliptic locus of some stratum  $\overline{\mathcal{M}}(m_1, \dots, m_k)$ , and denote by  $(d_1, \dots, d_n)$  the orders of singularities of corresponding quadratic differential. Then the sum of Lyapunov exponents of  $C$  is*

$$L(C) = \frac{1}{4} \sum_{j \text{ such that } d_j \text{ is odd}} \frac{1}{d_j+2}.$$

*Proof.* Because  $\sum_{i=1}^n d_i = -4$ , we have

$$\begin{aligned} L(C) &= 1 + \sum_{d_j \text{ odd}} \sum_{0 < 2k \leq d_j+1} \left(1 - \frac{2k}{d_j+2}\right) + \sum_{d_j \text{ even}} \sum_{0 < 2k \leq d_j+1} \left(1 - \frac{2k}{d_j+2}\right) \\ &= 1 + \sum_{0 < d_j \text{ odd}} \left(\frac{d_j}{4} + \frac{1}{4(d_j+2)}\right) + \sum_{d_j \text{ even}} \frac{d_j}{4} = 1 + \sum_{0 < d_j \text{ odd}} \frac{1}{4(d_j+2)} + \sum_{0 < d_j} \frac{d_j}{4} \\ &= 1 + \sum_{d_j \text{ odd}} \frac{1}{4(d_j+2)} + \sum_{d_j} \frac{d_j}{4} = \frac{1}{4} \sum_{d_j \text{ odd}} \frac{1}{d_j+2} \end{aligned}$$

□

TABLE 1. genus 3

| zeros     | component | Weierstrass exponents |       |            |
|-----------|-----------|-----------------------|-------|------------|
|           |           | $w_2$                 | $w_3$ | $\sum w_i$ |
| (4)       | hyp       | 3/5                   | 1/5   | 9/5        |
| (4)       | odd       | 2/5                   | 1/5   | 8/5        |
| (3,1)     |           | 2/4                   | 1/4   | 7/4        |
| (2,2)     | hyp       | 2/3                   | 1/3   | 2          |
| (2,2)     | odd       | 1/3                   | 1/3   | 5/3        |
| (2,1,1)   |           | 1/2                   | 1/3   | 11/6       |
| (1,1,1,1) |           |                       |       | varying    |

**5.3. The genus 3 case.** In what follows, the dimension of special linear systems have been discussed stratum by stratum in the corresponding section of [6]. Note that for a Teichmüller curve, by Equation (3), we have:

$$f_*\mathcal{O}(m_1D_1 + \dots + m_kD_k) = \mathcal{O}_C \oplus (f_*\mathcal{O}(m_1D_1 + \dots + m_kD_k)/\mathcal{O}_C).$$

In the stratum  $\overline{\Omega\mathcal{M}}_3(3,1)$ , a degenerating fibre is not hyperelliptic, so  $h^0(p_1 + p_2) = h^0(2p_1) = 1 = g - 2$ . By lemma 4.6, we have:

$$f_*\mathcal{O}(3D_1 + D_2) = \mathcal{O}_C \oplus \mathcal{O}_{D_1}(3D_1) \oplus \mathcal{O}_{D_2}(D_2).$$

In the stratum  $\overline{\Omega\mathcal{M}}_3^{odd}(2,2)$ , the theta characteristic is odd, so  $h^0(p_1 + p_2) = 1 = g - 2$ . By lemma 4.6, we have:

$$f_*\mathcal{O}(2D_1 + 2D_2) = \mathcal{O}_C \oplus \mathcal{O}_{D_1}(2D_1) \oplus \mathcal{O}_{D_2}(2D_2).$$

In the stratum  $\overline{\Omega\mathcal{M}}_3(2,1,1)$ , we have  $h^0(p_1 + p_3) = 1 = g - 2$ . By lemma 4.6, we have:

$$f_*\mathcal{O}(2D_1 + D_2 + D_3) = \mathcal{O}_C \oplus \mathcal{O}_{D_1}(2D_1) \oplus \mathcal{O}_{D_2}(D_2).$$

**5.4. The genus 4 case.** In the stratum  $\overline{\Omega\mathcal{M}}_4(5,1)$ , the curve  $C$  does not meet the pointed Brill-Noether divisor  $BN_{3,(2)}^1$ , so  $h^0(3p_1) = 1 = g - 3$ . By lemma 4.6:

$$f_*\mathcal{O}(5D_1 + D_2) = \mathcal{O}_C \oplus f_*\mathcal{O}_{2D_1}(5D_1) \oplus \mathcal{O}_{D_2}(D_2)$$

We also have  $h^0(4p_1) - 1 = h^0(3p_1) = h^0(2p_1) = 1$ , since by lemma 3.8:

$$f_*\mathcal{O}_{2D_1}(5D_1) = \mathcal{O}_{D_1}(5D_1) \oplus \mathcal{O}_{D_1}(4D_1).$$

In the stratum  $\overline{\Omega\mathcal{M}}_4^{even}(4,2)$ , we have  $h^0(2p_1 + p_2) = 2, h^0(3p_1 + 2p_2) = 3$ . If  $h^0(3p_1 + p_2) = 3$ , then by Riemann-Roch  $h^0(p_1 + p_2) = 2$ , hence  $p_1$  and  $p_2$  are in the same component of the fiber  $F$ . This component admits an involution that acts on the zeros of  $\omega$ . But  $p_1$  and  $p_2$  have different orders of zeros, so they cannot be conjugate by the involution. The contradiction implies that we have  $h^0(3p_1 + p_2) = 2, h^0(p_1 + p_2) = 1$ .

We get a Weierstrass pair filtration:

$$\{\{(1,1), (2,1)\}, \{(3,1), (3,2)\}, \{(3,2), (4,2)\}\}$$

$$(11) \quad f_*\mathcal{O}(1,1) \subset f_*\mathcal{O}(2D_1 + D_2) = f_*\mathcal{O}(3D_1 + D_2) \subset f_*\mathcal{O}(3D_1 + 2D_2) \subset f_*\mathcal{O}(4,2)$$

This is also the Harder-Narasimhan filtration, as the graded quotients are line bundles.

TABLE 2. genus 4

| zeros   | component | Weierstrass exponents |       |       |            |
|---------|-----------|-----------------------|-------|-------|------------|
|         |           | $w_2$                 | $w_3$ | $w_4$ | $\sum w_i$ |
| (6)     | hyp       | 5/7                   | 3/7   | 1/7   | 16/7       |
| (6)     | even      | 4/7                   | 2/7   | 1/7   | 14/7       |
| (6)     | odd       | 3/7                   | 2/7   | 1/7   | 13/7       |
| (5,1)   |           | 1/2                   | 2/6   | 1/6   | 2          |
| (3,3)   | hyp       | 3/4                   | 2/4   | 1/4   | 5/2        |
| (3,3)   | non-hyp   | 2/4                   | 1/4   | 1/4   | 2          |
| (4,2)   | even      | 3/5                   | 1/3   | 1/5   | 32/15      |
| (4,2)   | odd       | 2/5                   | 1/3   | 1/5   | 29/15      |
| (2,2,2) |           | 1/3                   | 1/3   | 1/3   | 2          |
| (3,2,1) |           | 1/2                   | 1/3   | 1/4   | 25/12      |

In the stratum  $\overline{\Omega\mathcal{M}}_4^{odd}(4,2)$ , the theta characteristic is odd, so  $h^0(2p_1 + p_2) = 1 = g - 3$ . By lemma 4.6:

$$f_*\mathcal{O}(4D_1 + 2D_2) = \mathcal{O}_C \oplus f_*\mathcal{O}_{2D_1}(4D_1) \oplus \mathcal{O}_{D_2}(2D_2)$$

We also have  $h^0(3p_1 + p_2) - 1 = h^0(2p_1 + p_2) = h^0(p_1 + p_2) = 1$ , since by lemma 3.8:

$$f_*\mathcal{O}_{2D_1}(4D_1) = \mathcal{O}_{D_1}(4D_1) \oplus \mathcal{O}_{D_1}(3D_1).$$

In the stratum  $\overline{\Omega\mathcal{M}}_4^{non-hyp}(3,3)$ , the curve  $C$  does not meet the pointed Brill-Noether divisor  $BN_{3,(1,1)}^1$ , so  $h^0(2p_1 + p_2) = 1 = g - 3$ . By lemma 4.6

$$f_*\mathcal{O}(3D_1 + 3D_2) = \mathcal{O}_C \oplus f_*\mathcal{O}_{D_1}(3D_1) \oplus \mathcal{O}_{2D_2}(3D_2)$$

We have  $h^0(3p_1 + p_2) - 1 = h^0(2p_1 + p_2) = h^0(p_1 + p_2) = 1$ , since by lemma 3.8:

$$f_*\mathcal{O}_{2D_2}(3D_2) = \mathcal{O}_{D_2}(3D_2) \oplus \mathcal{O}_{D_2}(2D_2)$$

In the stratum  $\overline{\Omega\mathcal{M}}_4^{odd}(2,2,2)$ , by Clifford's theorem, we get  $h^0(p_1 + p_2 + p_3) = 1$ . By lemma 4.6 we get

$$f_*\mathcal{O}(2D_1 + 2D_2 + 2D_3) = \mathcal{O}_C \oplus \mathcal{O}_{D_1}(2D_1) \oplus \mathcal{O}_{D_2}(2D_2) \oplus \mathcal{O}_{D_3}(2D_3).$$

In the stratum  $\overline{\Omega\mathcal{M}}_4(3,2,1)$ , the curve  $C$  does not meet the pointed Brill-Noether divisor  $BN_{4,(1,1,2)}^1$ , as it has been shown in [6] that  $h^0(2p_1 + p_2) = h^0(p_1 + p_2 + p_3) = 1 = g - 3$ . By lemma 4.6:

$$f_*\mathcal{O}(3D_1 + 2D_2 + D_3) = \mathcal{O}_C \oplus f_*\mathcal{O}_{2D_1}(3D_1) \oplus \mathcal{O}_{D_2}(2D_2)$$

We have  $h^0(3p_1 + p_2) - 1 = h^0(2p_1 + p_2) = h^0(p_1 + p_2) = 1$ , since by lemma 3.8

$$f_*\mathcal{O}_{2D_1}(3D_1) = \mathcal{O}_{D_1}(3D_1) \oplus \mathcal{O}_{D_1}(2D_1).$$

**5.5. The genus 5 case.** In the stratum  $\overline{\Omega\mathcal{M}}_5(5,3)$ , the curve  $C$  does not meet the pointed Brill-Noether divisor  $BN_{4,(1,2)}^1$ , therefore  $h^0(2p_1 + 2p_2) = 1$ , and by Riemann-Roch theorem, we get  $h^0(3p_1 + p_2) = 1 = g - 4$ . By lemma 4.6:

$$f_*\mathcal{O}(5D_1 + 3D_2) = \mathcal{O}_C \oplus \mathcal{O}_{2D_1}(5D_1) \oplus \mathcal{O}_{2D_2}(3D_2)$$

TABLE 3. genus 5

| zeros | component | Weierstrass exponents |       |       |       |            |
|-------|-----------|-----------------------|-------|-------|-------|------------|
|       |           | $w_2$                 | $w_3$ | $w_4$ | $w_5$ | $\sum w_i$ |
| (8)   | hyp       | 7/9                   | 5/9   | 3/9   | 1/9   | 25/9       |
| (8)   | even      | 5/9                   | 3/9   | 2/9   | 1/9   | 20/9       |
| (8)   | odd       | 4/9                   | 3/9   | 2/9   | 1/9   | 19/9       |
| (5,3) |           | 1/2                   | 1/3   | 1/4   | 1/6   | 9/4        |
| (6,2) | odd       | 3/7                   | 1/3   | 2/7   | 1/7   | 46/21      |
| (4,4) | hyp       | 4/5                   | 3/5   | 2/5   | 1/5   | 3          |

Thus  $h^0(4p_1 + p_2) - 1 = h^0(3p_1 + p_2) = h^0(2p_1 + p_2) = 1$ , and  $h^0(p_1 + 4p_2) - 1 = h^0(p_1 + 3p_2) = h^0(p_1 + 2p_2) = 1$ , so by lemma 3.8:

$$f_*\mathcal{O}_{2D_1}(5D_1) = \mathcal{O}_{D_1}(5D_1) \oplus \mathcal{O}_{D_1}(4D_1)$$

$$f_*\mathcal{O}_{2D_2}(3D_2) = \mathcal{O}_{D_2}(3D_2) \oplus \mathcal{O}_{D_2}(2D_2)$$

In the stratum  $\overline{\Omega\mathcal{M}}_5^{odd}(6,2)$ , we need the following lemma, the proof of which is due to D.Chen.

**Lemma 5.7** (Chen). *In the stratum  $\overline{\Omega\mathcal{M}}_5^{odd}(6,2)$ , we have  $h^0(3p_1 + p_2) = 1$ .*

*Proof.* If the curve  $X$  is smooth or irreducible, then  $h^0(3p_1 + p_2) = 3$  would imply, by Clifford's theorem, that  $X$  is hyperelliptic and  $2p_1$  linear equivalence to  $p_1 + p_2$  is impossible.

If  $X$  is reducible, it could have at most two components  $Z, Y$  meeting at  $n$  nodes ( $n > 1$ ), such that  $6 = 2g_1 - 2 + n$  and  $2 = 2g_2 - 2 + n$ . Therefore the only possibilities for  $(g_1, g_2, n)$  are  $(3, 1, 2)$  and  $(2, 0, 4)$ .

For the first case, the elliptic component  $Y$  contains  $p_2$  and  $h^0(p_2) = 1$ , i.e. all the sections are given by constant functions. The other component  $Z$  contains  $p_1$  with  $h^0(3p_1) < 3$ , and a section on  $Z$  uniquely determines the constant section on  $Y$ , by its values at the nodes (assuming the same value). Hence  $h^0(3p_1 + p_2) < 3$  on  $X$ .

For the second case,  $h^0(p_2) = 2$  on the rational component  $Y$ . Then  $h^0(3p_1)$  has to be 2 on  $Z$ , hence  $p_1$  is a Weierstrass point. But in order to glue two sections on  $Y$  and  $Z$ , they need to have the same value at each of the four nodes. The four nodes form two conjugate pairs on the hyperelliptic curve  $Z$ , hence gluing two sections still imposes two conditions. Therefore  $h^0(3p_1 + p_2) \leq 2 + 2 - 2 < 3$  on  $X$ .

Of course one has to check that the spin parity holds for nodal curves as well, i.e.  $h^0(3p_1 + p_2)$  cannot be two on a degenerate  $X$ . But this should follow from the well-defined odd/even spin moduli spaces.  $\square$

We get  $h^0(3p_1 + p_2) = 1 = g - 4$ . By lemma 4.6:

$$f_*\mathcal{O}(6D_1 + 2D_2) = \mathcal{O}_C \oplus \mathcal{O}_{3D_1}(6D_1) \oplus \mathcal{O}_{D_2}(2D_2)$$

and we have  $h^0(4p_1 + p_2) - 1 = h^0(3p_1 + p_2) = h^0(2p_1 + p_2) = 1$ , since by lemma 3.8:

$$f_*\mathcal{O}_{3D_1}(6D_1) = \mathcal{O}_{D_1}(6D_1) \oplus \mathcal{O}_{D_1}(5D_1) \oplus \mathcal{O}_{D_1}(4D_1).$$

**Theorem 5.8.** *A Teichmüller curve in the strata*

$$\begin{aligned} & \overline{\Omega\mathcal{M}}_3(3,1), \overline{\Omega\mathcal{M}}_3^{odd}(2,2), \overline{\Omega\mathcal{M}}_3(2,1,1) \\ & \overline{\Omega\mathcal{M}}_4(5,1), \overline{\Omega\mathcal{M}}_4^{odd}(4,2), \overline{\Omega\mathcal{M}}_4^{non-hyp}(3,3), \overline{\Omega\mathcal{M}}_4^{odd}(2,2,2), \overline{\Omega\mathcal{M}}_4(3,2,1) \\ & \overline{\Omega\mathcal{M}}_5(5,3), \overline{\Omega\mathcal{M}}_5^{odd}(6,2) \end{aligned}$$

*has explicit Weierstrass exponents given in Table 1, Table 2, Table 3, and  $f_*\omega_{S/C}$  splits into direct sum of line bundles.*

*A Teichmüller curve in the stratum  $\overline{\Omega\mathcal{M}}_4^{even}(4,2)$  has explicit Weierstrass exponents.*

*Proof.* In all case, we have constructed a filtration (10). □

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UNIVERSITÄT MAINZ, FACHBEREICH 17, MATHEMATIK, 55099 MAINZ, GERMANY  
*E-mail address:* yuf@uni-mainz.de, vvyufei@gmail.com

UNIVERSITÄT MAINZ, FACHBEREICH 17, MATHEMATIK, 55099 MAINZ, GERMANY  
*E-mail address:* zuok@uni-mainz.de